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1977 J. Phys. A: Math. Gen. 10 L113

(http://iopscience.iop.org/0305-4470/10/6/004)

J. Phys. A: Math. Gen., Vol. 10, No. 6, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

Number of spanning trees on a lattice†

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Received 28 March 1977

Abstract. The number of spanning trees on a large lattice is evaluated exactly for the square, triangular and honeycomb lattices.

A spanning tree of a lattice $\mathscr{L}$ is a graph drawn on $\mathscr{L}$ which connects all lattice sites and contains no polygons. For a regular lattice of N sites, the number of spanning trees on $\mathscr{L}$, T_N, behaves as e^{zN} for large N. We report here the exact values of z for the square (SQ), triangular (TR) and honeycomb (HC) lattices.

More specifically let

$$z = \lim_{N\to\infty} N^{-1} \ln T_N. \tag{1}$$

We find

$$\begin{aligned} z_{\mathrm{SQ}} &= \frac{4}{\pi}(1-3^{-2}+5^{-2}-7^{-2}+\ldots) = 1{\cdot}166\,243\,6\ldots \\ z_{\mathrm{TR}} &= \frac{3\sqrt{3}}{\pi}(1-5^{-2}+7^{-2}-11^{-2}+13^{-2}-\ldots) = 1{\cdot}615\,329\,68\ldots \\ z_{\mathrm{HC}} &= \tfrac{1}{2}z_{\mathrm{TR}} = 0{\cdot}807\,664\,8\ldots. \end{aligned} \tag{2}$$

It was first pointed out by Fortuin and Kasteleyn (1972) that T_N is expressible in terms of the partition function of a lattice statistical model. For our purpose, it suffices to consider the following graph generating function on $\mathscr{L}$:

$$Z_N(q, v) = \sum_G q^n v^e. \tag{3}$$

Here the summation extends over all graphs G on $\mathscr{L}$; n and e are, respectively, the numbers of clusters and edges in G. Z_N is proportional to the cluster generating function of Fortuin and Kasteleyn (1972), and coincides with the partition function of a q-component Potts model for integral q (Baxter 1973). Now let $v = q^\alpha$ and consider the $q \to 0$ limit of the function

$$Z_N(q, q^\alpha) = q^{\alpha N} \sum_G q^{\alpha c + (1-\alpha)n} \tag{4}$$

† Work supported in part by NSF Grant No. DMR 76-20643.

where we have used the Euler relation $N+c=n+e$ to eliminate the parameter e in favour of c, the number of independent circuits in G. For $\alpha=1$ the leading terms in (4) in the $q\to 0$ limit are the tree graphs ($c=0$). Consequently, $Z_N(q,q)$ generates forests of trees on $\mathscr{L}$ (Stephen 1976). For $0<\alpha<1$ the leading terms are the spanning trees ($c=0$, $n=1$). Thus we have the *exact* relation valid for any finite lattice

$$T_N=\lim_{q\to 0} q^{\alpha(1-N)-1} Z_N(q,q^{\alpha}), \qquad 0<\alpha<1. \tag{5}$$

Equation (5) reduces to (7.13) of Fortuin and Kasteleyn (1972) upon taking $\alpha=\frac{1}{2}$.

For planar $\mathscr{L}$ the generating function (3) is related to the partition function of an ice-type problem on a related medial lattice $\mathscr{L}'$ (Baxter *et al* 1976). The choice of $\alpha=\frac{1}{2}$ in (5) is especially convenient, for the resulting ice-type model is well defined and soluble in the $q\to 0$ limit. Combining (5) with (14) of Baxter *et al* (1976), we obtain (with $\alpha=\frac{1}{2}$) from (1)

$$z=\lim_{N\to\infty} N^{-1}\ln Z' \tag{6}$$

where Z' is the partition function of an ice-type model defined on the medial lattice $\mathscr{L}'$. If $\mathscr{L}$ is a square lattice, then $\mathscr{L}'$ is also a square lattice but having $2N$ sites. The vertex weights of the ice-type model on $\mathscr{L}'$ are (cf figure 5 of Baxter *et al* 1976)

$$\omega_1,\ldots,\omega_6=1,1,1,1,\sqrt{2},\sqrt{2}. \tag{7}$$

If $\mathscr{L}$ is a triangular (honeycomb) lattice, then $\mathscr{L}'$ is a Kagomé lattice of $3N$ ($3N/2$) sites with the following vertex weights:

$$\omega_1,\ldots,\omega_6=1,1,1,1,\mathrm{e}^{-\mathrm{i}\pi/6}+\mathrm{e}^{\mathrm{i}\pi/3},\mathrm{e}^{\mathrm{i}\pi/6}+\mathrm{e}^{-\mathrm{i}\pi/3}. \tag{8}$$

In either case, it is readily verified that the weights satisfy the free-fermion conditions (Fan and Wu 1970)

$$\omega_1\omega_2+\omega_3\omega_4=\omega_5\omega_6 \tag{9}$$

so that the right-hand side of (6) can be evaluated by computing a Pfaffian. In the case of square lattice, the free-fermion solution of Z' was first evaluated by Wu and reported in Lieb (1967). The numerical value (2) for z_{SQ} now follows from (20) of Lieb (1967) and the fact that $\mathscr{L}'$ contains $2N$ sites. In the case of Kagomé lattice, the free-fermion solution of Z' has been obtained by Lin (1975)†. In the notation shown in figure 2 of Lin (1975), we may rewrite (8) as

$$\begin{aligned}
&\omega_i=\omega_i'=\omega_i'', \qquad i=1,2,3,4\\
&\omega_5=\omega_5'=\omega_6''=\mathrm{e}^{-\mathrm{i}\pi/6}+\mathrm{e}^{\mathrm{i}\pi/3}\\
&\omega_6=\omega_6'=\omega_5''=\mathrm{e}^{\mathrm{i}\pi/6}+\mathrm{e}^{-\mathrm{i}\pi/3}
\end{aligned} \tag{10}$$

Equation (11) of Lin (1975)† now leads to

$$z_{\mathrm{TR}}=2z_{\mathrm{HC}}=\frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\ln[6-2\cos\theta-2\cos\phi-2\cos(\theta+\phi)]. \tag{11}$$

This reduces to (2) upon carrying out the integrations.

† The free energy given by Lin (1975) contains an error. The right-hand side of his equation (11) (and all other expressions for ϕ) should be multiplied by a factor $\frac{2}{3}$.

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